

MINIMAL SURFACES IN 4-DIMENSIONAL LORENTZIAN DAMEK-RICCI SPACES

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ABSTRACT. In this paper we will construct a Weierstrass type representation for minimal surfaces in 4-dimensional Lorentzian Damek-Ricci spaces and we give some examples of such surfaces.

1. INTRODUCTION

Damek-Ricci spaces are semidirect products of Heisenberg groups with the real line. They were considered in [2] (see also [1]), equipped with a left-invariant Riemannian metric, to give a negative answer, in high dimensions, to the question posed by Lichnerowicz: “is a harmonic Riemannian manifold necessarily a symmetric space?”

Beside a left-invariant Riemannian metric, these spaces may be equipped with left-invariant Lorentzian metrics in essentially two ways: a Riemannian metric on the Heisenberg factor and a negative metric on the \mathbb{R} factor, or a Lorentzian metric in the Heisenberg factor and a positive metric on \mathbb{R} . The aim of this paper is to study a Weierstrass representation for simply connected minimal surfaces in these spaces, in dimension four.

In [5] the authors give a Weierstrass representation theorem for minimal surfaces in Riemannian manifolds. In [4] this representation has been extended for timelike and spacelike minimal surfaces in 3-dimensional Lorentzian manifolds. The results can be easily extended to the case of minimal surfaces in Lorentzian manifolds of higher dimension. Most of the applications and examples of these results are given for 3-dimensional ambient spaces. For higher dimension, there is an application of this formula for minimal surfaces in 4-dimensional Damek-Ricci spaces equipped with a left-invariant Riemannian metric (see [3]).

This paper is organized as follows: in Section 2 we describe the geometry of the Damek-Ricci spaces and, then, in the next section we discuss the extensions of the Weierstrass representation theorem for minimal surfaces in Riemannian and Lorentzian manifolds. In Sections 4 and 5 we adapt the Weierstrass representation to our situation. Finally we give examples of spacelike and timelike minimal surfaces in these spaces (in the 4-dimensional case).

2. THE GEOMETRY OF THE DAMEK-RICCI SPACES

2.1. The generalized Heisenberg group. Let b_m and z_n be real vector spaces of dimensions m and n respectively, and $\beta : b_m \times b_m \rightarrow z_n$ a skew-symmetric bilinear map. In the direct sum $\mathfrak{h}_{m+n} = b_m \oplus z_n$ we define the bracket

$$(1) \quad [U + X, V + Y] = \beta(U, V).$$

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This product defines a Lie algebra structure on \mathfrak{h}_{m+n} , whose center contains z_n .

We endow b_m with a positive inner product and z_n with a positive or Lorentzian inner product. We will denote by $\langle \cdot, \cdot \rangle_{\mathfrak{h}_{m+n}}$ the product metric. For $Z \in z_n$, we define $J_Z \in \text{End}(b_m)$ by

$$(2) \quad \langle J_Z U, V \rangle_{\mathfrak{h}_{m+n}} = \langle \beta(U, V), Z \rangle_{\mathfrak{h}_{m+n}},$$

for all $U, V \in b_m$ and $Z \in z_n$.

The Lie algebra \mathfrak{h}_{m+n} is called a *generalized Riemannian Heisenberg algebra* if the inner product in z_n is positive and

$$J_Z^2 = -\langle Z, Z \rangle_{\mathfrak{h}_{m+n}} \text{id}_{b_m},$$

for all $Z \in z_n$. The associated simply connected Lie group, with the left-invariant metric, is called a *generalized Riemannian Heisenberg group*.

The Lie algebra \mathfrak{h}_{m+n} is called a *generalized Lorentzian Heisenberg algebra* if the inner product in z_n is Lorentzian and

$$J_Z^2 = -\langle Z, Z \rangle_{\mathfrak{h}_{m+n}} \text{id}_{b_m}, \quad \text{if } Z \text{ is spacelike,}$$

$$J_Z^2 = \langle Z, Z \rangle_{\mathfrak{h}_{m+n}} \text{id}_{b_m}, \quad \text{if } Z \text{ is timelike.}$$

The associated simply connected Lie group, with the left-invariant metric, is called a *generalized Lorentzian Heisenberg group*.

2.2. Lorentzian Damek-Ricci spaces of the first kind. Take the direct sum $\mathfrak{s}_{m+n+1} = \mathfrak{h}_{m+n} \oplus \mathfrak{a}$, where \mathfrak{a} is a Lorentzian one-dimensional space and \mathfrak{h}_{m+n} is a generalized Riemannian Heisenberg algebra. A vector in \mathfrak{s}_{m+n+1} can be written in a unique way as $U + X + sA$, for some $U \in b_m$, $X \in z_n$, $s \in \mathbb{R}$ and a non zero fixed vector A in \mathfrak{a} .

Given $U + X + rA, V + Y + sA \in \mathfrak{s}_{m+n+1}$, we define

$$\langle U + X + rA, V + Y + sA \rangle = \langle U + X, V + Y \rangle_{\mathfrak{h}_{m+n}} - rs$$

and

$$[U + X + rA, V + Y + sA] = [U, V]_{\mathfrak{h}_{m+n}} + \frac{1}{2}rV - \frac{1}{2}sU + rY - sX.$$

Then $\langle \cdot, \cdot \rangle$ is a Lorentzian metric and $[\cdot, \cdot]$ is a Lie bracket in \mathfrak{s}_{m+n+1} . Therefore, \mathfrak{s}_{m+n+1} is a Lie algebra. Moreover, $\langle A, A \rangle = -1$.

Definition 2.1. The simply connected Lie group associated to \mathfrak{s}_{m+n+1} , endowed with the induced left-invariant Lorentzian metric, is called a *Lorentzian Damek-Ricci space of the first kind* and will be denoted by \mathbb{S}_{m+n+1}^1 .

The Levi-Civita connection ∇ of \mathbb{S}_{m+n+1}^1 is given by

$$\begin{aligned} \nabla_{V+Y+sA}(U + X + rA) = \\ -\frac{1}{2}\{J_Y U + J_X V + rV + [U, V] + 2rY + \langle U, V \rangle A + 2\langle X, Y \rangle A\}. \end{aligned}$$

2.3. Lorentzian Damek-Ricci spaces of the second kind. We consider again the direct sum $\mathfrak{s}_{m+n+1} = \mathfrak{h}_{m+n} \oplus \mathfrak{a}$, where now \mathfrak{a} is a Riemannian 1-dimensional space and \mathfrak{h}_{m+n} is a generalized Lorentzian Heisenberg algebra. The bracket is given as above and the metric is given by

$$\langle U + X + rA, V + Y + sA \rangle = \langle U + X, V + Y \rangle_{\mathfrak{h}_{m+n}} + rs.$$

Definition 2.2. The simply connected Lie group associated to \mathfrak{s}_{m+n+1} , endowed with the induced left-invariant Lorentzian metric, is called a *Lorentzian Damek-Ricci space of the second kind* and will be denoted by \mathbb{S}_{m+n+1}^{m+n} .

The Levi-Civita connection ∇ of \mathbb{S}_{m+n+1}^{m+n} is given by

$$\begin{aligned} \nabla_{V+Y+sA}(U+X+rA) = \\ -\frac{1}{2}\{J_Y U + J_X V + rV + [U, V] + 2rY - \langle U, V \rangle A - 2\langle X, Y \rangle A\}. \end{aligned}$$

3. THE WEIERSTRASS REPRESENTATION

The Weierstrass representation theorem is an important tool in the study of minimal surfaces in \mathbb{R}^n since it allows to bring in the powerful theory of holomorphic functions. The local version has been extended to the case of minimal surfaces in a Riemannian manifold in [5] and for Lorentzian manifolds in [4]. In this section we will briefly discuss such extensions. We start with the Riemannian case. Since the considerations are local, we can suppose that the ambient manifold is \mathbb{R}^n with a Riemannian metric $g = [g_{ij}]$.

Theorem 3.1. *Let $\Omega \subseteq \mathbb{C}$ be an open set and let $f : \Omega \rightarrow \mathbb{R}^n$ be a conformal minimal immersion. Let $\{u, v\}$ be conformal coordinates in Ω and $z = u + iv$. Consider the complex tangent vector*

$$\frac{\partial f}{\partial z} := \frac{1}{2}\left(\frac{\partial f}{\partial u} + i\frac{\partial f}{\partial v}\right),$$

where $i = \sqrt{-1}$. Let

$$\frac{\partial f}{\partial z} = \sum_{i=1}^n \phi_i \frac{\partial}{\partial x_i}.$$

Then

- (1) $\sum_{i,j=1}^n g_{ij} \phi_i \phi_j \neq 0$,
- (2) $\sum_{i,j=1}^n g_{ij} \phi_i \overline{\phi_j} = 0$,
- (3) $\frac{\partial \phi_i}{\partial \bar{z}} + \sum_{j,l=1}^n \Gamma_{jl}^i \phi_j \overline{\phi_l} = 0$,

where Γ_{jl}^i are the Christoffel symbols of g . Moreover, if Ω is simply connected, the functions

$$f_i := 2 \operatorname{Re} \int \phi_i dz$$

are well defined and define a conformal minimal immersion with complex tangent vector

$$\frac{\partial f}{\partial z} = \sum_{i=1}^n \phi_i \frac{\partial}{\partial x_i}.$$

Remark 3.2. In the Theorem 3.1 the first condition guarantees that f is an immersion, the second one that f is conformal and the last one that f is minimal.

Remark 3.3. The third condition is called the *harmonicity condition* since it just says that the tension field vanishes.

In the case that g is a Lorentzian metric the essential difference is the following: for spacelike surfaces (i.e. if f^*g is Riemannian) the statement is the same. For timelike surfaces (i.e. if f^*g is Lorentzian) the expression $\frac{\partial \phi_i}{\partial \bar{z}}$, as well as the conjugation, has to be understood in the Lorentz or *paracomplex* sense.

We recall that the algebra of *paracomplex numbers* is the algebra

$$\mathbb{L} = \{a + \tau b, a, b \in \mathbb{R}\},$$

where τ is an imaginary unit with $\tau^2 = 1$. The operations are the obvious ones and the set of zero divisors is the set

$$K = \{a \pm \tau a, a \in \mathbb{R}\}.$$

This algebra is isomorphic to $\mathbb{R} \oplus \mathbb{R}$ via the map

$$a + \tau b \mapsto \frac{1}{2}(a + b, a - b).$$

Paraconjugation and norm are defined as in the complex case and $z \in \mathbb{L} \setminus K$ is invertible with inverse $z^{-1} = \bar{z}/(z\bar{z})$.

The set \mathbb{L} has a natural topology as a 2-dimensional real vector space.

Definition 3.4. Let $\Omega \subseteq \mathbb{L}$ be an open set and $z_0 \in \Omega$. The \mathbb{L} -derivative of a function $f : \Omega \rightarrow \mathbb{L}$ at z_0 is defined by

$$f'(z_0) := \lim_{\substack{z \rightarrow z_0 \\ z - z_0 \in \mathbb{L} \setminus K \cup \{0\}}} \frac{f(z) - f(z_0)}{z - z_0},$$

if the limit exists. If $f'(z_0)$ exists, we will say that f is \mathbb{L} -differentiable at z_0 .

Remark 3.5. The condition of \mathbb{L} -differentiability is much less restrictive than the usual complex differentiability. For example, \mathbb{L} -differentiability at z_0 does not imply continuity at z_0 . However, \mathbb{L} -differentiability in an open set $\Omega \subset \mathbb{L}$ implies usual differentiability in Ω .

Introducing the paracomplex operators:

$$\frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial u} + \tau \frac{\partial}{\partial v} \right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial u} - \tau \frac{\partial}{\partial v} \right),$$

where $z = u + \tau v$, we have that a differentiable function $f : \Omega \rightarrow \mathbb{L}$ is \mathbb{L} -differentiable if and only if

$$(3) \quad \frac{\partial f}{\partial \bar{z}} = 0.$$

We observe that, writing $f(u, v) = a(u, v) + \tau b(u, v)$, $u + \tau v \in \Omega$, the condition (3) is equivalent to the para-Cauchy-Riemann equations:

$$\begin{cases} \frac{\partial a}{\partial u} = \frac{\partial b}{\partial v}, \\ \frac{\partial a}{\partial v} = \frac{\partial b}{\partial u}, \end{cases}$$

whose integrability conditions are given by the wave equations

$$a_{uu} - a_{vv} = 0 = b_{uu} - b_{vv}.$$

We observe that the harmonicity condition is a system of partial differential equations (really an integral differential equation, since the Γ 's must be computed along a solution). Hence, in general, it is quite hard to find explicit solutions. However, for certain ambient spaces, as the Lie groups, these equations are essentially equivalent to a system of partial differential equations with constant coefficients. We will comment now, briefly, the case where the ambient space is a Lie group. In what follows \mathbb{K} will denote either the complex numbers \mathbb{C} or the Lorentz numbers \mathbb{L} .

Let M be a n -dimensional Lie group endowed with a left-invariant Riemannian or Lorentzian metric and let $f : \Omega \subset \mathbb{K} \rightarrow M$ be a conformal minimal immersion, where $\Omega \subset \mathbb{K}$ is an open set. Let $\{e_1, e_2, \dots, e_n\}$ be a left-invariant orthonormal frame field, with e_1, \dots, e_{n-1} spacelike and e_n timelike if the metric is Lorentzian. We can write the (para)complex tangent field $\phi = \frac{\partial f}{\partial z}$ along f both in terms of local coordinates $\{x_1, x_2, \dots, x_n\}$ in M and, also, using the left-invariant vector fields. Hence, one has

$$\phi = \sum_{i=1}^n \phi_i \frac{\partial}{\partial x_i} = \sum_{i=1}^n \psi_i e_i,$$

where the functions ϕ_i and ψ_i are related by

$$(4) \quad \phi_i = \sum_{j=1}^n A_{ij} \psi_j,$$

where $A : \Omega \rightarrow GL(n, \mathbb{R})$ is a smooth map. In terms of the components ψ_i , the harmonicity condition can be written as

$$\frac{\partial \psi_k}{\partial \bar{z}} + \frac{1}{2} \sum_{i,j=1}^n L_{ij}^k \bar{\psi}_i \psi_j = 0,$$

where the symbols L_{ij}^k are defined by

$$\nabla_{e_i} e_j = \frac{1}{2} \sum_{k=1}^n L_{ij}^k e_k.$$

Consequently, in the case of n -dimensional Lie groups, the Theorem 3.1 may be rephrased as follows

Theorem 3.6. *Let M be a n -dimensional Lie group endowed with a left-invariant Lorentzian metric and let $\{e_1, e_2, \dots, e_n\}$ be a left-invariant orthonormal frame field. Let $f : \Omega \rightarrow M$ be a conformal minimal immersion, where $\Omega \subset \mathbb{K}$ is an open set. We denote by $\phi \in \Gamma(f^*TM \otimes \mathbb{K})$ the (para)complex tangent vector*

$$\phi = \frac{\partial f}{\partial z}.$$

Then, the components ψ_i , $i = 1, \dots, n$, of ϕ satisfy the following conditions:

- i) $|\psi_1|^2 + |\psi_2|^2 + \dots + |\psi_{n-1}|^2 - |\psi_n|^2 \neq 0$,
- ii) $\psi_1^2 + \psi_2^2 + \dots + \psi_{n-1}^2 - \psi_n^2 = 0$,

$$\text{iii)} \quad \frac{\partial \psi_k}{\partial \bar{z}} + \frac{1}{2} \sum_{i,j=1}^n L_{ij}^k \bar{\psi}_i \psi_j = 0.$$

Conversely, if $\Omega \subset \mathbb{K}$ is a simply connected domain and $\psi_k : \Omega \rightarrow \mathbb{K}$, $k = 1, \dots, n$, are (para)complex functions satisfying the conditions above, then the map $f : \Omega \rightarrow M$ which coordinates are given by

$$f_i = 2 \operatorname{Re} \int \sum_{j=1}^n A_{ij} \psi_j dz, \quad i = 1, \dots, n,$$

is a well-defined conformal minimal immersion.

4. THE WEIERSTRASS REPRESENTATION IN THE LORENTZIAN DAMEK-RICCI SPACES \mathbb{S}_4^1

We consider the 4-dimensional space \mathbb{S}_4^1 with global coordinates $\{x, y, z, t\}$. The left-invariant Lorentzian metric g is given by:

$$g = e^{-t} dx^2 + e^{-t} dy^2 + e^{-2t} (dz + \frac{c}{2}y dx - \frac{c}{2}x dy)^2 - dt^2,$$

where $c \in \mathbb{R}$. The Lie algebra \mathfrak{s}_4 of \mathbb{S}_4^1 has an orthonormal basis

$$e_1 = e^{\frac{t}{2}} \left(\frac{\partial}{\partial x} - \frac{c}{2} y \frac{\partial}{\partial z} \right), \quad e_2 = e^{\frac{t}{2}} \left(\frac{\partial}{\partial y} + \frac{c}{2} x \frac{\partial}{\partial z} \right), \quad e_3 = e^t \frac{\partial}{\partial z}, \quad e_4 = \frac{\partial}{\partial t},$$

where e_1, e_2, e_3 are spacelike and e_4 is timelike. The Lie brackets are given by

$$\begin{cases} [e_1, e_2] = c e_3, & [e_1, e_3] = 0, & [e_1, e_4] = -\frac{1}{2} e_1, \\ [e_2, e_3] = 0, & [e_2, e_4] = -\frac{1}{2} e_2, & [e_3, e_4] = -e_3. \end{cases}$$

The Levi-Civita connection is given by:

$$\begin{aligned} \nabla_{e_1} e_1 &= -\frac{1}{2} e_4, & \nabla_{e_1} e_2 &= \frac{c}{2} e_3, & \nabla_{e_1} e_3 &= -\frac{c}{2} e_2, & \nabla_{e_1} e_4 &= -\frac{1}{2} e_1, \\ \nabla_{e_2} e_1 &= -\frac{c}{2} e_3, & \nabla_{e_2} e_2 &= -\frac{1}{2} e_4, & \nabla_{e_2} e_3 &= \frac{c}{2} e_1, & \nabla_{e_2} e_4 &= -\frac{1}{2} e_2, \\ \nabla_{e_3} e_1 &= -\frac{c}{2} e_2, & \nabla_{e_3} e_2 &= \frac{c}{2} e_1, & \nabla_{e_3} e_3 &= -e_4, & \nabla_{e_3} e_4 &= -e_3, \\ \nabla_{e_4} e_1 &= \nabla_{e_4} e_2 = \nabla_{e_4} e_3 = \nabla_{e_4} e_4 = 0. \end{aligned}$$

Also, we have that the non zero L_{ij}^k are:

$$\begin{aligned} L_{11}^4 &= -1, & L_{12}^3 &= c, & L_{13}^2 &= -c, & L_{14}^1 &= -1, & L_{21}^3 &= -c, & L_{22}^4 &= -1, \\ L_{23}^1 &= c, & L_{24}^2 &= -1, & L_{31}^2 &= -c, & L_{32}^1 &= c, & L_{33}^4 &= -2, & L_{34}^3 &= -2. \end{aligned}$$

The matrix A defined in the previous section is

$$A = \begin{bmatrix} e^{\frac{t}{2}} & 0 & 0 & 0 \\ 0 & e^{\frac{t}{2}} & 0 & 0 \\ -\frac{c}{2} e^{\frac{t}{2}} y & \frac{c}{2} e^{\frac{t}{2}} x & e^t & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The harmonicity condition is given by the following system of PDEs:

$$(5) \quad \begin{cases} \frac{\partial \psi_1}{\partial \bar{z}} - \frac{1}{2} \bar{\psi}_1 \psi_4 + c \operatorname{Re}(\bar{\psi}_2 \psi_3) = 0, \\ \frac{\partial \psi_2}{\partial \bar{z}} - \frac{1}{2} \bar{\psi}_2 \psi_4 - c \operatorname{Re}(\bar{\psi}_1 \psi_3) = 0, \\ \frac{\partial \psi_3}{\partial \bar{z}} - \bar{\psi}_3 \psi_4 + \frac{c}{2} (\bar{\psi}_1 \psi_2 - \bar{\psi}_2 \psi_1) = 0, \\ \frac{\partial \psi_4}{\partial \bar{z}} - \frac{1}{2} (\bar{\psi}_1 \psi_1 + \bar{\psi}_2 \psi_2) - \bar{\psi}_3 \psi_3 = 0. \end{cases}$$

Then Theorem 3.6 takes the form

Theorem 4.1. *Let $\Omega \subseteq \mathbb{K}$ be an open set endowed of a (para)complex coordinates and $f : \Omega \rightarrow \mathbb{S}_4^1$ a conformal minimal immersion. Then, the components of the (para)complex tangent vector*

$$\phi = \frac{\partial f}{\partial z} = \sum_{i=1}^4 \psi_i e_i$$

satisfy the system (5) and the following conditions:

- i) $|\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2 - |\psi_4|^2 \neq 0$,
- ii) $\psi_1^2 + \psi_2^2 + \psi_3^2 - \psi_4^2 = 0$.

Conversely, if Ω is simply connected and $\psi_i : \Omega \rightarrow \mathbb{K}$, $i = 1, 2, 3, 4$, are functions satisfying the above conditions, then the map $f : \Omega \rightarrow \mathbb{S}_4^1$ with coordinates

$$f_i = 2 \operatorname{Re} \int \sum_{j=1}^4 A_{ij} \psi_j dz, \quad i = 1, 2, 3, 4,$$

defines a conformal minimal immersion in \mathbb{S}_4^1 .

We will give now some examples.

Example 4.2. Consider the paracomplex functions

$$\psi_1 = \frac{\tau}{u}, \quad \psi_2 = \psi_3 = 0, \quad \psi_4 = \frac{1}{u},$$

defined in the simply connected domain $\Omega = \{u + \tau v \in \mathbb{L} \mid u > 0\}$. We have that (5) and conditions i) and ii) of Theorem 4.1 are satisfied and, so, the map $f : \Omega \rightarrow \mathbb{S}_4^1$ given by

$$\begin{cases} f_1 = \frac{2(v - v_0)}{u_0}, \\ f_2 = k \in \mathbb{R}, \\ f_3 = -\frac{k_1(v - v_0)}{u_0}, \quad k_1 \in \mathbb{R}, \\ f_4 = 2 \ln \left(\frac{u}{u_0} \right), \end{cases}$$

is a conformal timelike minimal immersion, where $z_0 = u_0 + \tau v_0 \in \Omega$.

Example 4.3. The paracomplex functions

$$\psi_1 = \psi_2 = \frac{\tau}{\sqrt{2}u}, \quad \psi_3 = 0, \quad \psi_4 = \frac{1}{u}$$

defined in $\Omega = \{u + \tau v \in \mathbb{L} \mid u > 0\}$ satisfy the Theorem 4.1. Then, the map $f : \Omega \rightarrow \mathbb{S}_4^1$ with coordinates

$$\begin{cases} f_1 = -\frac{2(v - v_0)}{u_0}, \\ f_2 = -\frac{2(v - v_0)}{u_0}, \\ f_3 = k \in \mathbb{R}, \\ f_4 = 2 \ln \left(\frac{u}{u_0} \right), \end{cases}$$

is a conformal timelike minimal immersion, where $z_0 = u_0 + \tau v_0 \in \Omega$.

5. THE WEIERSTRASS REPRESENTATION IN THE LORENTZIAN DAMEK-RICCI SPACES \mathbb{S}_4^3

We consider the 4-dimensional Lorentzian Damek-Ricci space \mathbb{S}_4^3 with global coordinates $\{x, y, z, t\}$. The left-invariant Lorentzian metric g on \mathbb{S}_4^3 is given by:

$$g = e^{-t} dx^2 + e^{-t} dy^2 - e^{-2t} \left(dz + \frac{c}{2} y dx - \frac{c}{2} x dy \right)^2 + dt^2,$$

where $c \in \mathbb{R}$. The Lie algebra \mathfrak{s}_4 of \mathbb{S}_4^3 has the orthonormal basis

$$e_1 = e^{\frac{t}{2}} \left(\frac{\partial}{\partial x} - \frac{c y}{2} \frac{\partial}{\partial z} \right), \quad e_2 = e^{\frac{t}{2}} \left(\frac{\partial}{\partial y} + \frac{c x}{2} \frac{\partial}{\partial z} \right), \quad e_3 = e^t \frac{\partial}{\partial z}, \quad e_4 = \frac{\partial}{\partial t},$$

where e_1, e_2, e_4 are spacelike and e_3 is timelike. The Lie brackets are given by

$$\begin{aligned} [e_1, e_2] &= c e_3, & [e_1, e_3] &= 0, & [e_1, e_4] &= -\frac{1}{2} e_1, \\ [e_2, e_3] &= 0, & [e_2, e_4] &= -\frac{1}{2} e_2, & [e_3, e_4] &= -e_3. \end{aligned}$$

As the Levi-Civita connection is given by:

$$\begin{aligned} \nabla_{e_1} e_1 &= \frac{1}{2} e_4, & \nabla_{e_1} e_2 &= \frac{c}{2} e_3, & \nabla_{e_1} e_3 &= \frac{c}{2} e_2, & \nabla_{e_1} e_4 &= -\frac{1}{2} e_1, \\ \nabla_{e_2} e_1 &= -\frac{c}{2} e_3, & \nabla_{e_2} e_2 &= \frac{1}{2} e_4, & \nabla_{e_2} e_3 &= -\frac{c}{2} e_1, & \nabla_{e_2} e_4 &= -\frac{1}{2} e_2, \\ \nabla_{e_3} e_1 &= \frac{c}{2} e_2, & \nabla_{e_3} e_2 &= -\frac{c}{2} e_1, & \nabla_{e_3} e_3 &= -e_4, & \nabla_{e_3} e_4 &= -e_3, \\ \nabla_{e_4} e_1 &= \nabla_{e_4} e_2 = \nabla_{e_4} e_3 = \nabla_{e_4} e_4 = 0, \end{aligned}$$

then the non zero L_{ij}^k are

$$\begin{aligned} L_{11}^4 &= 1, & L_{12}^3 &= c, & L_{13}^2 &= c, & L_{14}^1 &= -1, & L_{21}^3 &= -c, & L_{22}^4 &= 1, \\ L_{23}^1 &= -c, & L_{24}^2 &= -1, & L_{31}^2 &= c, & L_{32}^1 &= -c, & L_{33}^4 &= -2, & L_{34}^3 &= -2. \end{aligned}$$

Also the matrix A is given by

$$A = \begin{bmatrix} e^{\frac{t}{2}} & 0 & 0 & 0 \\ 0 & e^{\frac{t}{2}} & 0 & 0 \\ -\frac{c}{2} e^{\frac{t}{2}} y & \frac{c}{2} e^{\frac{t}{2}} x & e^t & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The harmonicity condition becomes

$$(6) \quad \begin{cases} \frac{\partial \psi_1}{\partial \bar{z}} - \frac{1}{2} \bar{\psi}_1 \psi_4 - c \operatorname{Re}(\bar{\psi}_2 \psi_3) = 0, \\ \frac{\partial \psi_2}{\partial \bar{z}} - \frac{1}{2} \bar{\psi}_2 \psi_4 + c \operatorname{Re}(\bar{\psi}_1 \psi_3) = 0, \\ \frac{\partial \psi_3}{\partial \bar{z}} - \bar{\psi}_3 \psi_4 + \frac{c}{2} (\bar{\psi}_1 \psi_2 - \bar{\psi}_2 \psi_1) = 0, \\ \frac{\partial \psi_4}{\partial \bar{z}} + \frac{1}{2} (\bar{\psi}_1 \psi_1 + \bar{\psi}_2 \psi_2) - \bar{\psi}_3 \psi_3 = 0. \end{cases}$$

Therefore, the Weierstrass representation formula is given by the following

Theorem 5.1. *Let $\Omega \subseteq \mathbb{K}$ be an open set endowed of a (para)complex coordinate and $f : \Omega \rightarrow \mathbb{S}_4^3$ a conformal minimal immersion. Then, the components of the (para)complex tangent vector*

$$\phi = \frac{\partial f}{\partial z} = \sum_{i=1}^4 \psi_i e_i,$$

satisfy the system (6) and the following conditions:

- i) $|\psi_1|^2 + |\psi_2|^2 - |\psi_3|^2 + |\psi_4|^2 \neq 0$,
- ii) $\psi_1^2 + \psi_2^2 - \psi_3^2 + \psi_4^2 = 0$.

Conversely, if Ω is simply connected and $\psi_i : \Omega \rightarrow \mathbb{K}, i = 1, 2, 3, 4$, are functions satisfying the above conditions, then the map $f : \Omega \rightarrow \mathbb{S}_4^3$ with coordinates

$$f_i = 2 \operatorname{Re} \int \sum_{j=1}^4 A_{ij} \psi_j dz, \quad i = 1, 2, 3, 4,$$

is a conformal timelike (or spacelike) minimal immersion in \mathbb{S}_4^3 .

We will give now some examples.

Example 5.2. It easy to check that the complex functions

$$\psi_1 = \frac{i}{u}, \quad \psi_2 = \psi_3 = 0, \quad \psi_4 = \frac{1}{u},$$

defined in $\Omega = \{u + i v \in \mathbb{C} \mid u > 0\}$, satisfy (6) and the conditions i) and ii) of Theorem 5.1. So, the map $f : \Omega \rightarrow \mathbb{S}_4^3$ given by:

$$\begin{cases} f_1 = -\frac{2(v - v_0)}{u_0}, \\ f_2 = k \in \mathbb{R}, \\ f_3 = \frac{k_1(v - v_0)}{u_0}, \quad k_1 \in \mathbb{R}, \\ f_4 = 2 \ln \left(\frac{u}{u_0} \right), \end{cases}$$

is a conformal spacelike minimal immersion, where $z_0 = u_0 + i v_0 \in \Omega$.

Example 5.3. Consider the paracomplex functions

$$\psi_1 = \psi_2 = 0, \quad \psi_3 = \frac{\tau}{2u}, \quad \psi_4 = \frac{1}{2u}$$

in the simply connected domain $\Omega = \{u + \tau v \in \mathbb{L} \mid u > 0\}$.

We have that they satisfy (6) and the conditions i) and ii) of Theorem 5.1. Then, the map $f : \Omega \rightarrow \mathbb{S}_4^3$ with components:

$$\begin{cases} f_1 = k_1 \in \mathbb{R}, \\ f_2 = k_2 \in \mathbb{R}, \\ f_3 = \frac{(v - v_0)}{u_0}, \\ f_4 = \ln \left(\frac{u}{u_0} \right), \end{cases}$$

is a conformal timelike minimal immersion in \mathbb{S}_4^3 , where $z_0 = u_0 + \tau v_0 \in \Omega$.

REFERENCES

- [1] J. Berndt, F. Tricerri, L. Vanhecke. *Generalized Heisenberg Groups and Damek-Ricci Harmonic Spaces*, Lecture Notes in Mathematics, **1598**, Springer-Verlag, Berlin, (1991).
- [2] E. Damek, F. Ricci. *A class of nonsymmetric harmonic Riemannian spaces*, Bull. Amer. Math. Soc. **27** (1992), 139-142.
- [3] M. Koivogui, L. Todjihounde. *Weierstrass Representation for minimal immersions into Damek-Ricci spaces*, Int. Electron. J. Geom. **6** (2013), 1-7.
- [4] J.H. Lira, M. Melo, F. Mercuri. *A Weierstrass representation for minimal surfaces in 3-dimensional manifolds*, Results. Math. **60** (2011), 311-323.
- [5] F. Mercuri, S. Montaldo, P. Piu. *A Weierstrass representation formula of minimal surfaces in \mathbb{H}_3 and $\mathbb{H}^2 \times \mathbb{R}$* , Acta Math. Sinica **22** (2006), 1603-1612.

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